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LETTERS TO THE EDITOR

ON THE ATTAINED WAITING TIME

J. W. COHEN*, *Centre for Mathematics and Computer Science*

Abstract

By using properties of up- and downcrossings of the sample functions of the workload process and of the attained waiting-time process for a $G/G/1$ queueing model, a direct proof of a theorem proved by Sakasegawa and Wolff is given.

WORKLOAD PROCESS; SAMPLE FUNCTIONS; STATIONARY DISTRIBUTIONS

Sakasegawa and Wolff (1990) show by using sample function arguments that for the FIFO $G/G/1$ queueing model the workload process $\boldsymbol{v}_t$ and the attained waiting-time process $\boldsymbol{\eta}_t$ possess the same stationary distribution, if such distributions exist. However their proof is somewhat artificial (see their use of preemptive LIFO).

A direct proof of their Theorem 1 proceeds as follows. Consider a busy cycle $\boldsymbol{c}$ with $\boldsymbol{n}$ the number of customers served; $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_n$ are the service times of these customers, $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n$ their successive actual waiting times, $\boldsymbol{i}$ the idle time, so

$$\boldsymbol{c} = \boldsymbol{\tau}_1 + \cdots + \boldsymbol{\tau}_n + \boldsymbol{i}. \tag{1}$$

The attained service time $\boldsymbol{\eta}_t$ at epoch t is by definition the time between t and the arrival epoch of the customer being served at epoch t. In Figure 1 the sample function of the workload process $\boldsymbol{v}_t$ and the corresponding $\boldsymbol{\eta}_t$-process during the busy cycle $\boldsymbol{c}$ are shown, with $\boldsymbol{n} = 4$.

Define for $v \geqq 0$,

$$\begin{aligned} \boldsymbol{d}(v) &:= \#\text{ downcrossings of } \boldsymbol{v}_t,\ 0 \leqq t \leqq \boldsymbol{c} \text{ with level } v,\ (*) \\ \boldsymbol{u}(v) &:= \#\text{ upcrossings of } \boldsymbol{v}_t,\ 0 \leqq t \leqq \boldsymbol{c} \text{ with level } v,\ (^\circ) \end{aligned} \tag{2}$$

$$\begin{aligned} \delta(v) &:= \#\text{ upcrossings of } \boldsymbol{\eta}_t,\ 0 \leqq t \leqq \boldsymbol{c} \text{ with level } v,\ (*) \\ \omega(v) &:= \#\text{ downcrossings of } \boldsymbol{\eta}_t,\ 0 \leqq t \leqq \boldsymbol{c} \text{ with level } v,\ (^\circ). \end{aligned} \tag{3}$$

Note that in the figure $\boldsymbol{d}(v) = 3$; the upcrossings are there indicated by °, the downcrossings by *. It is immediately evident from the geometry of the sample functions (see Cohen (1977), (1982)) that with probability 1, for $v \geqq 0$,

$$\boldsymbol{d}(v) = \boldsymbol{u}(v), \qquad \boldsymbol{\delta}(v) = \boldsymbol{\omega}(v), \tag{4}$$

$$\boldsymbol{u}(v) = \boldsymbol{\delta}(v); \tag{5}$$

and

$$\boldsymbol{d}(v) = \frac{d}{dv} \int_0^{\boldsymbol{c}} (\boldsymbol{v}_t < v)\, dt, \qquad \boldsymbol{\delta}(v) = \frac{d}{dv} \int_0^{\boldsymbol{c}} (\boldsymbol{\eta}_t < v)\, dt, \tag{6}$$

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* Postal address: Stichting Mathematisch Centrum, Centre for Mathematics and Computer Science, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands.

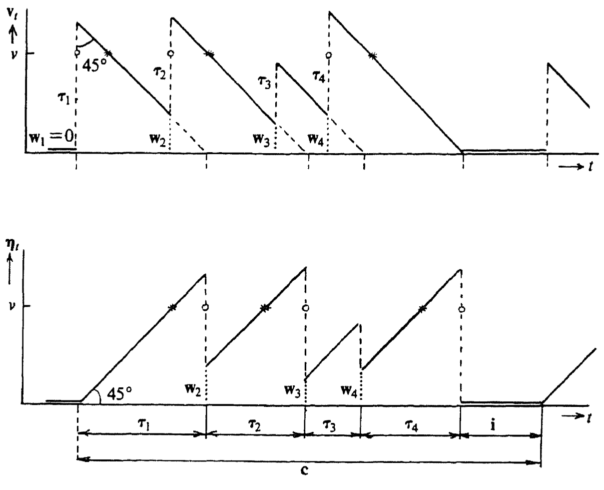

Figure 1

where we use the notation

(7) $$(\boldsymbol{v}_t < v) \equiv 1_{\boldsymbol{v}_t < v} \quad \text{and} \quad \int_0^c (\boldsymbol{v}_t < v)\, dt \equiv \int_0^\infty (\boldsymbol{v}_t < v,\ \boldsymbol{c} \geqq t)\, dt,$$

for the indicator function and the integral. Since

(8) $$\boldsymbol{i} = \left\{ \int_0^c (\boldsymbol{v}_t < v)\, dt \right\}_{v=0+} = \left\{ \int_0^c (\boldsymbol{\eta}_t < v)\, dt \right\}_{v=0+},$$

integration of (6), using the boundary conditions (8) yields, via (4) and (5), that with probability 1

(9) $$\int_0^c (\boldsymbol{v}_t < v)\, dt = \int_0^c (\boldsymbol{\eta}_t < v)\, dt, \qquad v \geqq 0.$$

Because

$$(\boldsymbol{v}_t < v) = 1 - (\boldsymbol{v}_t \geqq v),$$

we have from (9)

$$\int_0^c (\boldsymbol{v}_t \geqq v)\, dt = \int_0^c (\boldsymbol{\eta}_t \geqq v)\, dt,$$

which is Theorem 1 of Sakasegawa and Wolff (1990).

References

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